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水素原子の波動関数を求める。

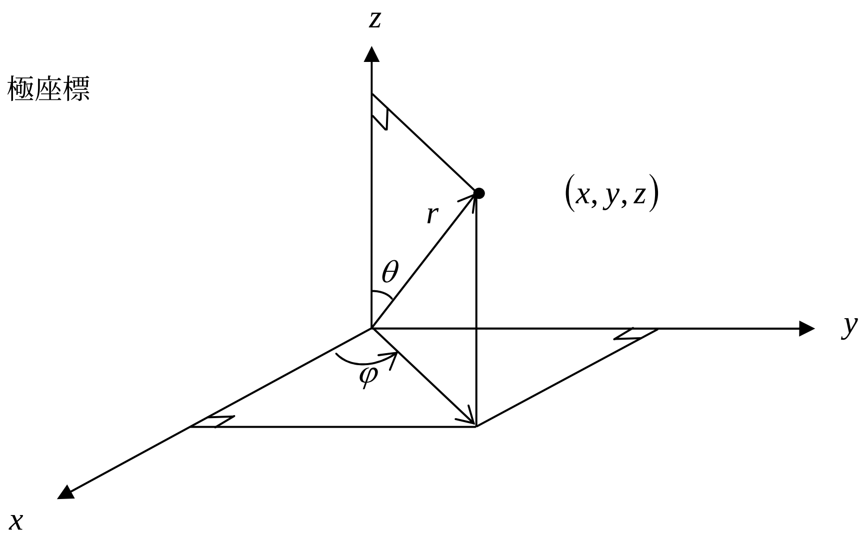
質量ｍの粒子がポテンシャル$V(r)$の力場内にあるとき，Schrodinger方程式は

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(r)u = Eu$$ である。ここで,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ は*Laplacian*を表わす。

まず，ラプラシアンを極座標で表示する。極座標は,

極座標



$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

$$\varphi = \tan^{-1}\frac{y}{x}$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

ここで，$f(r,\theta,\varphi)$ という関数を考える。

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x}$$

$$\therefore \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

同様に

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}+\frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi}$$

となる。この時,

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\sin\theta\cos\varphi$$

$$\frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta\sin\varphi$$

$$\frac{\partial r}{\partial z}=\frac{z}{r}=\cos\theta$$

また，$y=\tan^{-1}x,\ x=\tan y=\dfrac{\sin y}{\cos y}$ であるから

$$\frac{dx}{dy}=\frac{\cos^2 y+\sin^2 y}{\cos^2 y}=\frac{1}{\cos^2 y}$$

$$\frac{dy}{dx}=\cos^2 y$$

$$\frac{d\tan\theta}{d\theta}=\frac{d}{d\theta}\cdot\frac{\sin\theta}{\cos\theta}=\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\frac{\sqrt{x^2+y^2}}{z}=\frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta}=\frac{x^2+y^2}{z^2}=\frac{1-\cos^2\theta}{\cos^2\theta}$$

$$\cos^2\theta\left(x^2+y^2\right)=z^2\left(1-\cos^2\theta\right)$$

$$\therefore\cos^2\theta=\frac{z^2}{x^2+y^2+z^2}$$

$\theta=\tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z}$ $\left(\dfrac{\sqrt{x^2+y^2}}{z}=\tan\theta\right)$ より,

$$\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial(\tan\theta)}\cdot\frac{\partial(\tan\theta)}{\partial x}=\cos^2\theta\cdot\frac{\partial\dfrac{\sqrt{x^2+y^2}}{z}}{\partial\left(x^2+y^2\right)}\cdot\frac{\partial\left(x^2+y^2\right)}{\partial x}$$

$$= \cos^2\theta \cdot \frac{1}{z} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

$$= \frac{z^2}{x^2+y^2+z^2} \cdot \frac{1}{z} \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{r^2} \cdot \frac{x}{\sqrt{x^2+y^2}} \cdot z = \frac{x}{r} \cdot \frac{1}{r} \cdot \frac{z}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{r}\sin\theta\cos\varphi\frac{1}{\tan\theta} = \frac{1}{r}\cos\theta\cos\varphi$$

$$\frac{\partial\theta}{\partial y} = \cos^2\theta \cdot \frac{1}{z} \cdot \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} \cdot \frac{1}{r}\frac{z}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{r}\sin\theta \cdot \sin\varphi \cdot \frac{1}{\tan\theta} = \frac{1}{r}\cos\theta\sin\varphi$$

$$\frac{\partial\theta}{\partial z} = \cos^2\theta \cdot \frac{\partial}{\partial z} \cdot \left(\frac{\sqrt{x^2+y^2}}{z}\right) = \frac{z^2}{x^2+y^2+z^2} \cdot \sqrt{x^2+y^2} \cdot \left(-\frac{1}{z^2}\right)$$

$$= -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} = -\frac{z}{r^2} \cdot \frac{\sqrt{x^2+y^2}}{z} = -\frac{1}{r}\cos\theta \cdot \tan\theta = -\frac{1}{r}\sin\theta$$

となる。一方,

$$\varphi = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial\tan\varphi} \cdot \frac{\partial\tan\varphi}{\partial x} = \cos^2\varphi \cdot \left(-\frac{y}{x^2}\right)$$

$$= -\cos^2\varphi \cdot \frac{y}{x} \cdot \frac{1}{x} = -\cos^2\varphi \cdot \tan\varphi \cdot \frac{1}{r\sin\theta\cos\varphi}$$

$$= -\cos^2\varphi \cdot \frac{\sin\varphi}{\cos\varphi} \cdot \frac{1}{r\sin\theta\cos\varphi} = -\frac{\sin\varphi}{r\sin\theta}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\tan\varphi} \cdot \frac{\partial\tan\varphi}{\partial y} = \cos^2\varphi \cdot \frac{1}{x}$$

$$= \cos^2\varphi \cdot \frac{1}{r\sin\theta \cdot \cos\varphi} = \frac{\cos\varphi}{r\sin\theta}$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial\tan\varphi} \cdot \frac{\partial\tan\varphi}{\partial z} = \phi$$

$$\left(\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\right)=-\frac{\cos\theta}{\sin^2\theta}\right)$$

よって,

$$\frac{\partial}{\partial x}=\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y}=\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

次に

$$\frac{\partial^2}{\partial x^2}=\sin\theta\cos\varphi\frac{\partial}{\partial r}\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$=\sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta}$$

$$+\frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi\sin\varphi}{r^2}\frac{\partial}{\partial\varphi}$$

$$-\frac{\sin\varphi\cos\varphi}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial r}$$

$$+\frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}-\frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta}$$

$$+\frac{1}{r^2}\cos^2\theta\cos^2\varphi\frac{\partial^2}{\partial\theta^2}+\frac{\cos^2\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}$$

$$-\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}+\frac{\sin\theta\sin^2\varphi}{r\sin\theta}\frac{\partial}{\partial r}$$

$$-\frac{\sin\varphi\cos\varphi}{r}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial r}+\frac{\cos\theta\sin^2\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$-\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\theta}+\frac{\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}+\frac{\sin^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\frac{\partial^2}{\partial y^2}=\sin\theta\sin\varphi\frac{\partial}{\partial r}\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$=\sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}$$

$$+\frac{1}{r}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\frac{\sin\theta\sin\varphi\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$+\frac{\sin\theta\sin\varphi\cos\varphi}{r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\cos^2\theta\sin^2\varphi\frac{\partial}{\partial r}$$

$$+\frac{1}{r}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}-\frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}$$

$$+\frac{1}{r^2}\cos^2\theta\sin^2\varphi\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\frac{\cos^2\theta\sin\varphi\cos\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi}$$

$$+\frac{1}{r^2}\frac{\cos\theta\sin\varphi\cos\varphi}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}+\frac{\sin\theta}{r\sin\theta}\cos^2\varphi\frac{\partial}{\partial r}$$

$$+\frac{\sin\theta}{r\sin\theta}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}\frac{\partial}{\partial r}+\frac{\cos\theta\cos^2\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$+\frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\theta}-\frac{\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}$$

$$+\frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\frac{\partial^2}{\partial z^2}=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}$$

$$+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

以上の項をまとめると

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$=\sin^2\theta\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r}\sin\theta\cos\theta\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}$$
$$+\frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}+\frac{1}{r}\sin\theta\cos\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}-\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}$$
$$+\frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}$$
$$+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$+\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}$$
$$+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}$$
$$+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$
$$=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

これで，ラプラシアンを極座標に変換出来た事になる。
従って，Schrodinger方程式は

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}+\frac{2mr^2}{\hbar^2}[E-V(r)]u=\phi$$

となる。

この方程式を解くためにまず，変数分離を行う。
$u(r,\theta,\varphi)=R(r)Y(\theta,\varphi)$　とおく。

$$\frac{\partial}{\partial r}\left(r^2Y\frac{dR}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(R\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}R\frac{\partial^2 Y}{\partial\varphi^2}+\frac{2mr^2}{\hbar^2}[E-V(r)]RY=0$$

RYで割ると,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right)+\frac{2mr^2}{\hbar^2}\left[E-V(r)\right]=-\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right]$$

左右両辺は定数でなければならない。この定数をλで表わせば,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}\left[E-V(r)\right]-\frac{\lambda}{r^2}\right]R=0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+\lambda Y=0$$

更に,$Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$とする

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\Phi\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\Theta\frac{\partial^2\Phi}{\partial\varphi^2}+\lambda\Theta\Phi=0$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}+\lambda=0$$

$$\frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\sin^2\theta\cdot\lambda=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=\nu$$

とすると

$$\frac{d^2\Phi}{d\varphi^2}+\nu\Phi=0,$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\lambda-\frac{\nu}{\sin^2\theta}\right)\Theta=0$$

まず,関数$\Phi(\alpha)$を考えよう。

$$\frac{d^2\Phi}{d\varphi^2}+\nu\Phi=\phi \qquad ここで, \qquad \sqrt{\nu}=m \quad とおく。$$

この一般解は,

$\Phi=Ae^{im\varphi}+Be^{-im\varphi};\nu\neq 0$

$\Phi=A+B\varphi \qquad ;\nu=0$

$\varphi=\phi$　と　$\varphi=2\pi$　で波は連続していなければならない。

$$A+B=Ae^{im2\pi}+Be^{-im2\pi} \qquad ――― \quad ①$$

又　$\varphi=0$　と　$\varphi=2\pi$　波動関数がなめらかにつながらなければならない(微分係数が等しい。)という条件から,

$$imA\left(e^0\right)-imB\left(e^0\right)=imAe^{im\cdot 2\pi}-imBe^{-im2\pi}$$

$$A-B=Ae^{im2\pi}-Be^{-im2\pi} \qquad ――― \quad ②$$

①+②　　$A=Ae^{i2m\pi}$

$$\therefore e^{im2\pi}=1$$

$$m=\cdots,-3,-2,-1,0,1,2,3,\cdots \qquad 整数$$

AとBをひとつにまとめると

$$\Phi = Ae^{im\varphi} \quad (m = 0, \pm1, \pm2, \pm3, \cdots)$$

次に規格化を行う。

確率密度関数$\Phi^*\Phi$を0から2πまで積分した値が１でなくてはならない。

$$\int_0^{2\pi} A^* e^{-im\varphi} \cdot Ae^{im\varphi} d\varphi = 1$$

$$\int_0^{2\pi} A^* A d\varphi = 1$$

$$A^* A[\varphi]_0^{2\pi} = 1$$

$$A^2 \cdot 2\pi = 1$$

Aを正の実数とすると

(波動関数に$e^{i\theta}$をかけたψは常に同じ解)

$$A = \frac{1}{\sqrt{2\pi}}$$

$$\therefore \Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \qquad m = 0,\ \pm1,\ \pm2,\ \cdots$$

ここで，mを磁気量子数という。

次に，$\Theta(\theta), R(r)$に関しては，それぞれ，球面調和関数及びLaguerreの陪関数という特殊関数で表わされることが分かっている。

$$u_{n,l,m}(r,\theta,\varphi) = \left[\left(\frac{2Z\mu}{na_0 m_e} \right)^{2l+3} \frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \frac{(n-l-1)!}{2n[(n+l)!]^3} \right]^{1/2} \times r^l e^{-\frac{Z\mu}{na_0 m_e} r} L_{n+l}^{2l+1}\left(\frac{2Z\mu}{na_0 m_e} r \right) P_l^{|m|}(\cos\theta) e^{im\varphi}$$

n : 量子数（principal quantum number), $\quad n = 1,2,3,\cdots$

l : 方位量子数（azimuthal quantum number), $\quad 0 \leqq l \leqq n-1$

m : 磁気量子数（magnetic quantum number), $\quad -l \leqq m \leqq l$

$u_{n,l,m}(r,\theta,\varphi)$はHamiltonian　$\hat{H}$，角運動量の２乗$\hat{l}^2$，角運動量のz成分$\hat{l}_z$の同時の固有関数で

$$\hat{H}u_{n,l,m}(r,\theta,\varphi) = E_n u_{n,l,m}(r,\theta,\varphi),$$
$$\hat{l}^2 u_{n,l,m}(r,\theta,\varphi) = l(l+1)\hbar^2 u_{n,l,m}(r,\theta,\varphi),$$
$$\hat{l}_z u_{n,l,m}(r,\theta,\varphi) = m\hbar u_{n,l,m}(r,\theta,\varphi).$$

詳細については，下記の書物を参照。

原島　鮮著．初等量子力学　裳華房S.47年.